In trying to explain this discrepancy, it was noted (see Ref. 15 for details) that most of the data considered in Ref. 6 was either in the core or just downstream of the core region. Asymptotic decay exponents cannot be meaningfully defined in the transition region where m ranges from zero, just at the end of the core, to some finite value further downstream; therefore, values of m obtained from data for this region cannot be interpreted as meaningful asymptotic decay exponents.

Conclusions

The uncertainty involved in calculating the centerline mass fraction decay exponent as demonstrated from the preceding discussion leads to the following conclusions. Existing data does not support use of Eq. (1) to characterize centerline mass fraction decay with a universal decay exponent. Also the centerline mass fraction decay exponent cannot be correlated with the velocity ratio U_j/U_e or the mass flux ratio $(\rho U)_j/(\rho U)_e$ using existing experimental data. If such a correlation exists, data for the far downstream region covering a wide range of velocity and density ratios must be made available. Two facts which make such data difficult to obtain are: 1) in the far downstream regions the parameters being measured are small and hence subject to large experimental uncertainties, and 2) the simulation of an infinite external stream becomes increasingly difficult far downstream due to external influences, e.g., walls. Therefore, to obtain such data the use of superior diagnostic techniques and more sophisticated experimental facilities than used to date are required.

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Estimation of Turbulent Energy Dissipation Using Available Transfer Theories

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Introduction

THE closure problem of the turbulent kinetic energy equation in physical space requires the specification of the dissipation function ε as a function of the energy

$$\bar{e} = \sum_{i=1}^{3} \overline{u_i u_i}.$$

On dimensional considerations, Rotta[1] proposed the relation

$$\varepsilon = A(R)(\bar{e})^{3/2}/L \tag{1}$$

where L is the integral scale of the turbulence and $R = (\varepsilon\nu)^{1/4}L/\nu$. For sufficiently high Reynolds numbers, the function $A(R)$ becomes a universal constant. This constant value has been used by many authors to obtain solutions of turbulent boundary layer problems.‡ Near the wall, however, the value of R and Re become small and a constant value for $A(R)$ is not valid ($Re = (\bar{e})^{1/2}L/\nu$). Direct use of the energy equation near the wall requires a nonconstant formulation for $A(R)$.

Recently Trusov[4] obtained an expression which provides the necessary values of $A(R)$ based on consideration of the three-dimensional turbulent-energy spectral density function $E(k)$, where k is the wave-number, in the universal equilibrium range. Trusov shows impressive agreement with a wide variety of test data down to fairly low values of Reynolds number.§ Unfortunately, some of the test data used by Trusov do not provide measured values of the integral scale so that these data, as used by Trusov, may be subject to question. The deduced expression for $E(k)$ used by Trusov is essentially the same as that used by Pao.[5] Pao also provided comparative figures using a variety of experimental results. Reid[6] has also provided some comparative results using the formulations of the spectral density functions proposed by Heisenberg, Obukhoff and Kovasznay. Reid used the experimental data of Stewart and Townsend.[7]

The objective of this Note is to evaluate the previously referenced formulations of the turbulent energy spectral density function, including variations on Pao's expression, to obtain the function that seems to provide the best fit with the available test data over the full range of Reynolds numbers. The criteria used to evaluate the validity of the various density functions are a combination of those used by the previously referenced authors plus one additional consideration. The additional consideration is that the resulting values of $A(R)$ should approach a value of about 0.16 as the Reynolds number becomes very large. The value of 0.16 represents an average value not greatly different from the value used by Bradshaw[8] (0.1643), the value measured in wall turbulence by Lawn[9] (0.125), and the results of Rose[10] (0.17). This value is significantly less than the value of 0.314 obtained from the expression used by Beckwith and Bushnell.[11]

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Index category: Boundary Layers and Convective Heat Transfer—Turbulent.

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‡ There are a number of extensive bibliographies available on the subject of turbulent boundary-layer flows. For the present Note only two more recent review papers are pointed out, Refs. 2 and 3.

§ The range of Reynolds numbers shown in Ref. 4 extend from $R = 3$ ($Re \cong 4$) to $R = 500$ ($Re \cong 7630$).

Analysis

Following the approach of Reid,[6] the energy density functions referred to in the previous section can be arranged in the following form

$$E_J(k) = C_J \alpha_J^{-3/2} \varepsilon^{1/4} \nu^{5/4} F_J(x_J) \tag{2}$$

where

$$x_J = \eta k/\beta_J$$

The Kolomogrov length scale η is defined as $\eta = (\nu^3/\varepsilon)^{1/4}$, and the subscripts denote the author (i.e., $J = H$ represents Heisenberg's theory, $J = K$ represents Kovasznay's theory, etc.). The appropriate relationships are defined as follows:

Heisenberg

$$F_H(x_H) = (2^{13}/3^7)^{1/4} x_H^{-5/3}(1 + x_H^4)^{-4/3}$$

$$\beta_H = (3\alpha_H^2/8)^{1/4}, \quad C_H = 1.0, \quad \alpha_H = 0.6$$

Kovasznay

$$F_K(x_K) = 2^{1/4} x_K^{-5/3}(1 - x_K^{4/3})^2/8$$

$$\beta_K = (32\alpha_K^2)^{1/4}, \quad C_K = 1.0, \quad \alpha_K = 0.318$$

Pao and Trusov

$$F_J(x_J) = x_J^{-5/3} \exp(-3\alpha_J^3 x_J^{4/3}/2)$$

$$\beta_J = \alpha_J^{3/2}, \quad C_P = 1.0, \quad C_T = 0.3, \quad \alpha_P = 1.7, \quad \alpha_T = 5.0$$

The relationships obtained with Obukhoff's theory are not presented in this Note due to space limitations and the fact that the numerical results did not provide a reasonable match with test data. Some discussion of this theory is presented in the latter part of this Note. In addition to these theories, the present authors consider Pao's expression with $C_P = 0.75$ and $\alpha_P = 2.0$.

Integrating the density functions over the wave number space provides the kinetic energy of the turbulence per unit mass. To circumvent the problem associated with defining $E(k)$ for small values of k, the approach of Trusov is used wherein the density function is assumed to be constant and equal to $E(k_S)$ for $0 \leqq k \leqq k_S$. The wave number is assumed to be equal to $1/L$ for the purpose of this study. As used, this wave number is representative of the turbulent energy containing eddies and could be considered a measure of the lower limit of wave numbers associated with the inertial exchange process. Integrating the density function in this manner yields the following unified expression for the energy:

$$\begin{aligned} \bar{e} &= k_S E_J(k_S) + \int_{k_S}^{\infty} E_J(k)\, dk \\ &= D_J R^{2/3} (\varepsilon\nu)^{1/2} f_J(R) \end{aligned} \tag{3}$$

Using an upper cutoff of 1.0 for the Kovasznay theory, the following relationships define the terms appearing in Eq. (3). For the Heisenberg, Kovasznay, and Pao and Trusov theories

$$f_J(R) = \lambda_J(x_{JS}) + x_{JS}^{2/3} \int_{x_{JS}}^{\gamma_J} x_J^{-5/4} \lambda_J(x_J)\, dx_J \tag{4}$$

where

$$D_H = 4(9\alpha_H)^{-2/3}, \quad \lambda_H(x_H) = (1 + x_H^4)^{-4/3}, \quad \gamma_H = \infty$$

$$D_K = (2\alpha_K)^{-2/3}, \quad \lambda_K(x_K) = (1 - x_K^{4/3})^2, \quad \gamma_K = 1$$

$$D_P = C_P \alpha_P, \quad \lambda_P(x_P) = \exp(-3\alpha_P^3 x_P^{4/3}/2), \quad \gamma_P = \infty$$

$$D_T = C_T \alpha_T, \quad \lambda_T(x_T) = \exp(-3\alpha_T^3 x_T^{4/3}/2), \quad \gamma_T = \infty$$

Manipulation of the equations yields the following additional relationships:

$$\begin{aligned} (\bar{e})^2/\varepsilon\nu &= [D_J f_J(R)]^2 R^{4/3} \\ A(R) &= [D_J f_J(R)]^{-3/2} \\ Re &= [D_J f_J(R)]^{1/2} R^{4/3} \end{aligned} \tag{5}$$

The one-dimensional spectrum functions used for comparing the various density function formulations are obtained from the three-dimensional density functions by the equation

$$\phi(k_S) = \frac{1}{2} \int_{k_S}^{\infty} \left[\frac{(1 - k_S^2/k^2)}{k}\right] E(k)\, dk$$

which can be considered in the form

$$\phi_J(k_S) = C_J \alpha_J^{-3/2} \varepsilon^{1/4} \nu^{5/4} \psi_J(x_{JS}) \tag{6}$$

where

$$\psi_J(x_{JS}) = \frac{1}{2} \int_{x_{JS}}^{\infty} \left[\frac{(1 - x_{JS}^2/x_J^2)}{x_J}\right] F_J(x_J)\, dx_J$$

In terms of the $\partial^3 u_1/\partial x_1^3$ fluctuations

$$(k_S \eta)^6 \phi(k_S)/\varepsilon^{1/4}\nu^{5/4} = C_J \alpha_J^{-3/2} \beta_J^6 x_{JS}^6 \psi_J(x_{JS}) \tag{7}$$

Results and Discussion

Numerical evaluation of the functions defined in Eqs. (5–7) was made for all the transfer theories considered in this Note. The results are discussed in the following paragraphs.

The values of $A(R)$ computed for each theory indicate that the limiting values¶ of $A(R)$, as Re becomes large, are 0.175 and 0.167 for the theories of Heisenberg and Kovasznay, respectively. Considering $C_P = 1.0$, Pao's theory yields a limiting value of 0.12 for $\alpha_P = 1.7$ and 0.09 for $\alpha_P = 2.2$. The limiting value of $A(R)$ for $C_P = 0.75$ and $\alpha_P = 2.0$ is equal to 0.16 which is essentially that used by Bradshaw.[8] The limiting value of $A(R)$ is 0.54 with the theory of Obukhoff and 0.15 with the theory of Trusov.

A comparison between the various theories and experimental data, represented by Trusov's derived results, is shown in terms of $\varepsilon\nu/(\bar{e})^2$ in Fig. 1. Obukhoff's theory displays a different trend with R [consistent with the higher limiting value of $A(R)$] and is not given further consideration. The other theories show fair agreement with the results of Trusov.

In Fig. 2 the one-dimensional spectrum function of $\partial^3 u_1/\partial x_1^3$ is shown for the various theories compared to the test data of Stewart and Townsend.[7] The test data are for $x_1/M \geqq 60$, and the results of Heisenberg and Kovaszany are not shown because Reid's paper adequately illustrates the inability of these formulations to match the experimental data for $k_S \eta > 0.7$. Except for low values of $k_S \eta$ (higher values of R), the theory of Trusov is seen to be in complete disagreement with the test data. Pao's theory with $C_P = 1.0$ and $\alpha_P = 2.2$ fits the test data well, but the limiting value of $A(R)$ is low. Pao recommended $\alpha_P = 1.7$, but

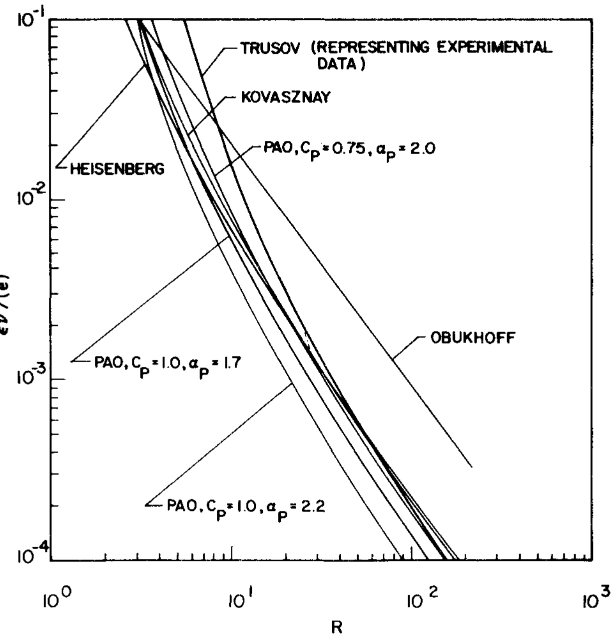

Fig. 1 Variation of $\varepsilon\nu/(\bar{e})^2$ with R.

¶ The value of $A(R)$ for $Re \sim 5000$. Numerical values of $A(R)$ for $Re \sim 10^9$ showed a slight decrease for all the transfer theories.

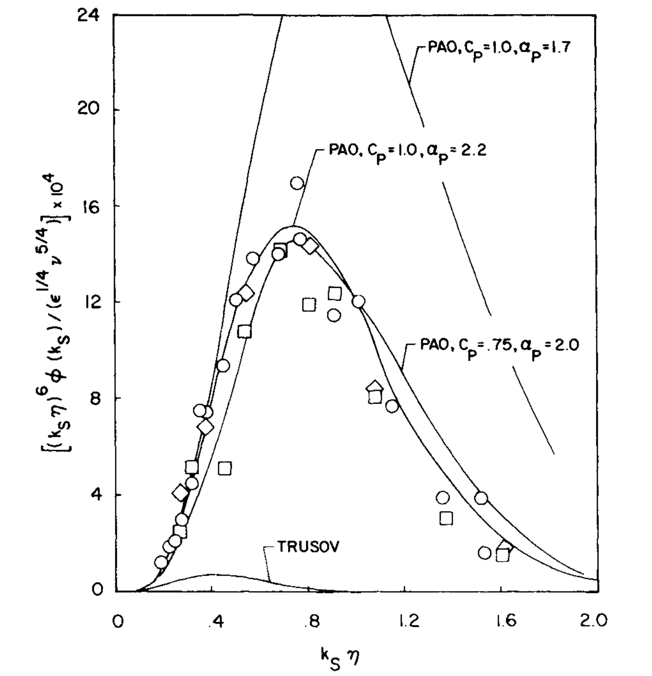

Fig. 2 Variation of the spectrum function of $\partial^3 u_1/\partial x_1{}^3$ fluctuations with wave number.

the derived results do not match the test results at the higher values of $k_S\eta$. Using $C_P = 0.75$ and $\alpha_P = 2.0$ gives fair agreement while giving the desired limiting value of $A(R)$.

The one-dimensional energy spectrum functions for Trusov's theory and Pao's theory with $C_P = 0.75$ and $\alpha_P = 2.0$ are compared with the test data of Ref. 5, represented by Pao's results for $C_P = 1.0$ and $\alpha_P = 1.7$, in Fig. 3. Trusov's theory again shows differences from the experimental data.

Based on these results it is felt that the use of Pao's density function with $C_P = 0.75$ and $\alpha_P = 2.0$ provides the best over-all match with the various forms of the available test data. Considering the variation of $A(R)$ with Re obtained with this density function, the following relations are obtained:

$$A(R) = 0.16 + 3.54/Re, \quad \text{for} \quad Re \geqq 1.0$$
$$A(R) = 0.16 + 3.54/(Re)^2, \quad \text{for} \quad 0.5 < Re \leqq 1.0$$

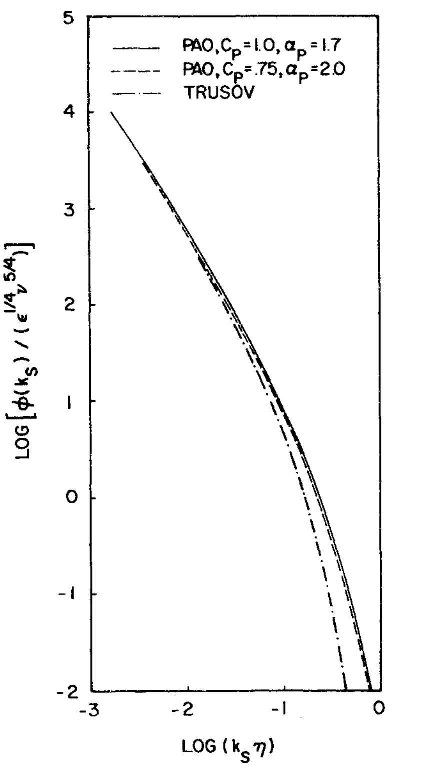

Fig. 3 Variations of the spectrum function of u_1 fluctuations with wave number.

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Edge Restraint Effect on Vibration of Curved Panels

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Introduction

In recent years, investigations have shown that edge restraint effects are quite important in the buckling of shell structures. The compressive buckling of cylindrical panels with various limiting cases of edge restraint along the unloaded edges has been studied by Rehfield and Hallauer.[1] A less exhaustive study of cylindrical panels under external pressure has been done by Singer et al.[2] These results indicate a high degree of edge restraint sensitivity for buckling and prompt us to investigate this effect on vibration in this Note.

Some attention has been paid to edge restraint effects on vibration. Forsberg[3] studied complete homogeneous isotropic cylindrical shells and found a particularly pronounced effect

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